Theory and Tool Support for the Formal Verification of Cryptographic Protocols

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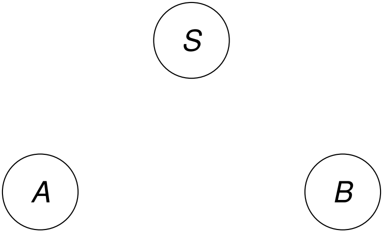
Cryptographic protocols are error-prone

Cryptographic protocols

To secure communication over insecure networks (e.g. Internet).
A communication protocol that uses *cryptography* to achieve security goals.

The Yahalom protocol



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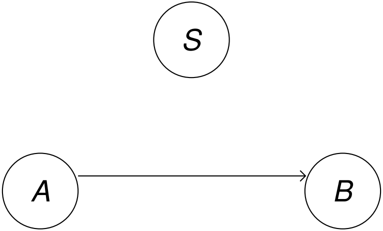
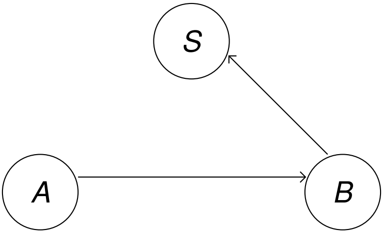


1. $A \rightarrow B : \quad (A \,.\, n_A)$

The Yahalom protocol



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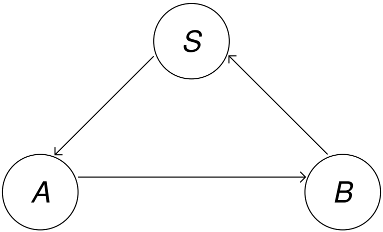

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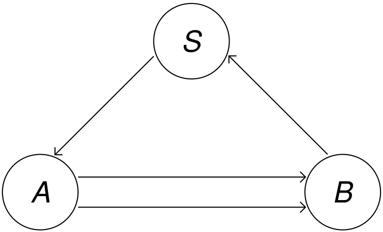

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... are error-prone

- Even when assuming perfect cryptographic primitives
- Canonical example: Needham-Schroeder with public key

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Why is it difficult?

Distributed algorithms that have the obligation to behave robustly in the context of unknown hostile attackers.

The spi calculus approach
Abadi and Gordon

- Cryptographic protocols are described in a precise and concise way.

- Equations to formulate security objectives.
  - secrecy: $P\{^M/_x\} \approx P\{^N/_x\}$ for any M and N
  - authenticity

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$$
\begin{array}{l}
(\nu k_{AS}, k_{BS}) \\
\quad\quad (\nu n_A)\, \overline{B}\langle (A \,.\, n_A) \rangle . A(x_2).\phi_2 \overline{B}\langle E_2 \rangle . \mathbf{0} \\
\quad \mid (\nu n_B)\, B(x_0).\phi_0 \overline{S}\langle (B \,.\, \mathrm{Enc}^{s}_{k_{BS}}(A \,.\, (\pi_2\,(x_0) \,.\, n_B))) \rangle . B(x_3).\phi_3\, \mathbf{0} \\
\quad \mid (\nu k_{AB})\, S(x_1).\phi_1 \overline{A}\langle E_1 \rangle . \mathbf{0}
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Testing equivalence

- Usually $\approx$ stands for *testing equivalence*.
- Intuitively, P and Q are testing equivalent *if and only if* they reveal the same information to observers (i.e. attackers).

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- Formally, P passes the test (R, β) iff $P \mid R \Downarrow_\beta$, i.e. $P \mid R$ may communicate on channel β.
- $P \simeq Q$ iff they pass the same tests, i.e. for any (R, β),

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- Problem: infinite quantification over arbitrary observers R.
- In practise, we define sound approximations that are easier to work with: bisimulations.

Bisimulations

- Behaviour of processes is described with a *Labelled Transitions System*: $P \xrightarrow{\mu} P'$
- Two processes are bisimilar if they can play the same transitions

$$\begin{tikzcd} P \arrow[d, dash] \\ Q \end{tikzcd}$$

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Q replies to P

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$$
\begin{tikzcd}
P \arrow[r, "\mu"] \arrow[d, no head] & P' \arrow[d, dashed, no head] \\
Q \arrow[r, dashed, "\mu"] & Q'
\end{tikzcd}
$$

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P \arrow[r, dashed, "\mu"] \arrow[d, dash] & P' \arrow[d, dash, dashed] \\
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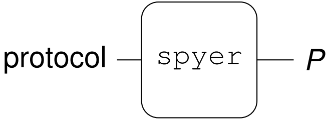
P replies to Q

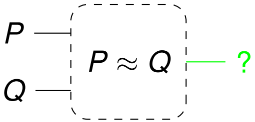
Contributions



From protocol narrations to spi calculus

A formal semantics for protocol narrations.
A rigorous translation to spi calculus.

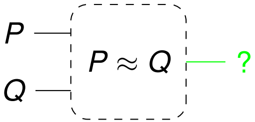
Contributions



Deciding process equivalence

A new notion of bisimulation for the spi calculus.
A symbolic characterisation.

Contributions



Towards a certified tool

Formalization of large parts of the developed theory in Coq.
Dream: Have a correct-by-construction tool.

Contributions

```
1 subgoal

============================
bisimilar P Q
```

Reasoning within Coq

Reason formally about cryptographic protocols in Coq.

Outline

1. From protocol narrations to spi calculus
2. An open variant of bisimulation for the spi calculus
3. A formalization in Coq

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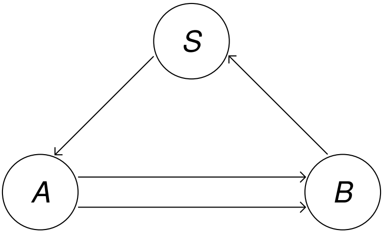

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The Yahalom protocol in spi-calculus

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\begin{array}{l}
(\nu k_{AS}, k_{BS}) \\
\quad (\nu n_A)\, \overline{B}\langle (A \,.\, n_A) \rangle . A(x_2) . \phi_2 \overline{B}\langle (\pi_2\,(x_2) \,.\, \mathrm{Enc}^{s}_{\pi_2\left(\pi_1\left(\mathrm{Dec}^{s}_{k_{AS}} \pi_1 (x_2)\right)\right)} \pi_2 \left(\pi_2 \left(\mathrm{Dec}^{s}_{k_{AS}} \pi_1\,(x_2)\right)\right)) \rangle .\, \mathbf{0} \\
\mid (\nu n_B)\, B(x_0) . \phi_0 \overline{S}\langle (B \,.\, \mathrm{Enc}^{s}_{k_{BS}}(A \,.\, (\pi_2\,(x_0) \,.\, n_B))) \rangle . B(x_3) . \phi_3\, \mathbf{0} \\
\mid (\nu k_{AB}) \\
\quad S(x_1) . \phi_1 \\
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\quad (\nu n_A)\, \overline{B}\langle (A \,.\, n_A)\rangle.A(x_2).\color{red}{\phi_2}\color{black}\overline{B}\langle(\pi_2\,(x_2)\,.\,\mathrm{Enc}^{s}_{\pi_2\left(\pi_1\left(\mathrm{Dec}^{s}_{k_{AS}}\pi_1(x_2)\right)\right)}\pi_2\left(\pi_2\left(\mathrm{Dec}^{s}_{k_{AS}}\pi_1\,(x_2)\right)\right))\rangle.\,\mathbf{0} \\
\mid (\nu n_B)\, B(x_0).\color{red}{\phi_0}\color{black}\overline{S}\langle(B\,.\,\mathrm{Enc}^{s}_{k_{BS}}(A\,.\,(\pi_2\,(x_0)\,.\,n_B)))\rangle.B(x_3).\color{red}{\phi_3}\color{black}\,\mathbf{0} \\
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\end{array}
$$

$$
\begin{array}{lcl}
\phi_0 & = & [A = \pi_1\,(x_0)] \\
\phi_1 & = & [\pi_1\left(\pi_2\left(\mathrm{Dec}^{s}_{k_{BS}}\pi_2\,(x_1)\right)\right) : \boldsymbol{M}] \wedge [B = \pi_1\,(x_1)] \wedge [A = \pi_1\left(\mathrm{Dec}^{s}_{k_{BS}}\pi_2\,(x_1)\right)] \\
\phi_2 & = & [B = \pi_1\left(\pi_1\left(\mathrm{Dec}^{s}_{k_{AS}}\pi_1\,(x_2)\right)\right)] \wedge [n_A = \pi_1\left(\pi_2\left(\mathrm{Dec}^{s}_{k_{AS}}\pi_1\,(x_2)\right)\right)] \\
\phi_3 & = & [A = \pi_1\left(\mathrm{Dec}^{s}_{k_{BS}}\pi_1\,(x_3)\right)] \wedge [n_B = \mathrm{Dec}^{s}_{\pi_2\left(\mathrm{Dec}^{s}_{k_{BS}}\pi_1(x_3)\right)} \pi_2\,(x_3)]
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State explicitly the assumptions

A protocol narration does not explicitly state the initial knowledge and what is to be generated freshly during a protocol run.

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A, S \textbf{ share } k_{AS} \\
B, S \textbf{ share } k_{BS} \\
A \textbf{ generates } n_A \;;\; B \textbf{ generates } n_B \;;\; S \textbf{ generates } k_{AB} \;; \\
A \rightsquigarrow B : (A \,.\, n_A) \;; \\
B \rightsquigarrow S : (B \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, (n_A \,.\, n_B))) \;; \\
S \rightsquigarrow A : (\mathrm{Enc}^{\mathrm{s}}_{k_{AS}}((B \,.\, k_{AB}) \,.\, (n_A \,.\, n_B)) \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, k_{AB})) \;; \\
A \rightsquigarrow B : (\mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, k_{AB}) \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{AB}} n_B)
\end{array}
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Principals act concurrently

A protocol narration describes an idealised sequential trace of execution whereas the principals act concurrently.

$A \rightarrow B : M$ actually means

(i) A *asynchronously* sends M towards B,

(ii) B receives some message

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$A \rightarrow B : M$ actually means

(i) A *asynchronously* sends M towards B,

(ii) B receives some message (intended to be M)

Principals perform on-reception checks

(iii) B checks that the message it just received has the expected properties.

State explicitly the assumptions

A protocol narration does not explicitly state the initial knowledge and what is to be generated freshly during a protocol run.

A, S **share** k_{AS}
B, S **share** k_{BS}
A **generates** n_A ; B **generates** n_B ; S **generates** k_{AB} ;
$A \rightsquigarrow B : (A \,.\, n_A)$;
$B \rightsquigarrow S : (B \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, (n_A \,.\, n_B)))$;
$S \rightsquigarrow A : (\mathrm{Enc}^{\mathrm{s}}_{k_{AS}}((B \,.\, k_{AB}) \,.\, (n_A \,.\, n_B)) \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, k_{AB}))$;
$A \rightsquigarrow B : (\mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, k_{AB}) \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{AB}} n_B)$

Generating the checks

Current knowledge

$$\{A, B, S, k_{AS}, n_A\}$$

expected	
$(\mathrm{Enc}^{\mathrm{s}}_{k_{AS}}((B \,.\, k_{AB}) \,.\, (n_A \,.\, n_B)) \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, k_{AB}))$	

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Generating the checks

Current knowledge

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expected	*actual*
$(\mathrm{Enc}^{s}_{k_{AS}}((B \,.\, k_{AB}) \,.\, (n_A \,.\, n_B)) \,.\, \mathrm{Enc}^{s}_{k_{BS}}(A \,.\, k_{AB}))$	x

Generating the checks

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expected	*actual*
$(\mathrm{Enc}^{s}_{k_{AS}}((B \,.\, k_{AB}) \,.\, (n_A \,.\, n_B)) \,.\, \mathrm{Enc}^{s}_{k_{BS}}(A \,.\, k_{AB}))$	x
$\mathrm{Enc}^{s}_{k_{AS}}((B \,.\, k_{AB}) \,.\, (n_A \,.\, n_B))$	$\pi_1\,(x)$
$\mathrm{Enc}^{s}_{k_{BS}}(A \,.\, k_{AB})$	$\pi_2\,(x)$

Generating the checks

Current knowledge

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expected	*actual*
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Generating the checks

Current knowledge

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n_B	$\pi_2\left(\pi_2\left(\mathrm{Dec}^{s}_{k_{AS}}\pi_1\,(x)\right)\right)$

Generating the checks

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$((B\,.\,k_{AB})\,.\,(n_A\,.\,n_B))$	$\mathrm{Dec}^{\mathrm{s}}_{k_{AS}}\pi_1\,(x)$
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$(n_A\,.\,n_B)$	$\pi_2\left(\mathrm{Dec}^{\mathrm{s}}_{k_{AS}}\pi_1\,(x)\right)$
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n_B	$\pi_2\left(\pi_2\left(\mathrm{Dec}^{\mathrm{s}}_{k_{AS}}\pi_1\,(x)\right)\right)$

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\;|\, (\nu n_B)\, B(x_0).\phi_0 \overline{S}\langle (B \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, (\pi_2\,(x_0) \,.\, n_B)))\rangle . B(x_3).\phi_3\, \mathbf{0} \\
\;|\, (\nu k_{AB}) \\
\qquad S(x_1).\phi_1 \\
\qquad \overline{A}\langle (\mathrm{Enc}^{\mathrm{s}}_{k_{AS}}((B \,.\, k_{AB}) \,.\, (\pi_1\left(\pi_2\left(\mathrm{Dec}^{\mathrm{s}}_{k_{BS}}\pi_2\,(x_1)\right)\right) \,.\, \pi_2\left(\pi_2\left(\mathrm{Dec}^{\mathrm{s}}_{k_{BS}}\pi_2\,(x_1)\right)\right))) \,.\, \mathrm{Enc}^{\mathrm{s}}_{k_{BS}}(A \,.\, k_{AB}))\rangle . \mathbf{0}
\end{array}
$$

$$
\begin{array}{lcl}
\phi_0 & = & [A = \pi_1\,(x_0)] \\
\phi_1 & = & [\pi_1\left(\pi_2\left(\mathrm{Dec}^{\mathrm{s}}_{k_{BS}}\pi_2\,(x_1)\right)\right) : \boldsymbol{M}] \wedge [B = \pi_1\,(x_1)] \wedge [A = \pi_1\left(\mathrm{Dec}^{\mathrm{s}}_{k_{BS}}\pi_2\,(x_1)\right)] \\
\phi_2 & = & [B = \pi_1\left(\pi_1\left(\mathrm{Dec}^{\mathrm{s}}_{k_{AS}}\pi_1\,(x_2)\right)\right)] \wedge [n_A = \pi_1\left(\pi_2\left(\mathrm{Dec}^{\mathrm{s}}_{k_{AS}}\pi_1\,(x_2)\right)\right)] \\
\phi_3 & = & [A = \pi_1\left(\mathrm{Dec}^{\mathrm{s}}_{k_{BS}}\pi_1\,(x_3)\right)] \wedge [n_B = \mathrm{Dec}^{\mathrm{s}}_{\pi_2\left(\mathrm{Dec}^{\mathrm{s}}_{k_{BS}}\pi_1(x_3)\right)}\pi_2\,(x_3)]
\end{array}
$$

Outline

Situation in the pi calculus

- Spi calculus is an extension of the pi calculus that incorporates cryptographic primitives .

$$\begin{array}{rcl} P, Q & ::= & \mathbf{0} \mid a(x).P \mid \overline{a}\langle u\rangle.P \\ & \mid & [a=b]P \mid (\nu x)\,P \\ & \mid & P \mid Q \mid P+Q \mid\, !\,P \end{array}$$

- Open bisimulation (Sangiorgi) is at the basis of several tools that automatically checks equivalence of pi terms
  e.g. the Mobility Workbench (Victor)
- Can we extend this notion to the spi calculus?

Situation in the pi calculus

- Spi calculus is an extension of the pi calculus that incorporates cryptographic primitives more.

$$
\begin{array}{rcl}
P, Q & ::= & \mathbf{0} \mid E(x).P \mid \overline{E}\langle F\rangle.P \\
 & \mid & \phi P \mid (\nu x)\, P \\
 & \mid & P \,|\, Q \mid P + Q \mid\, !\, P
\end{array}
$$

$$
\begin{array}{rcl}
M, N & ::= & x \mid (M\,.\,N) \mid \mathrm{Enc}^{\mathrm{s}}_{N} M \\
E, F & ::= & \ldots \mid \pi_1\,(E) \mid \pi_2\,(E) \mid \mathrm{Dec}^{\mathrm{s}}_{F} E \\
\phi & ::= & [\,E = F\,] \mid [\,E : \mathcal{N}\,]
\end{array}
$$

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Situation in the pi calculus

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$$\begin{array}{rcl} P, Q & ::= & \mathbf{0} \mid a(x).P \mid \overline{a}\langle u \rangle.P \\ & \mid & [a=b]P \mid (\nu x)\, P \\ & \mid & P \,|\, Q \mid P + Q \mid \,!\, P \end{array}$$

- Open bisimulation (Sangiorgi) is at the basis of several tools that automatically checks equivalence of pi terms
  e.g. the Mobility Workbench (Victor)
- Can we extend this notion to the spi calculus?

Bisimulations in the pi calculus

The main differences is the way they handle substitutions

$$\begin{tikzcd} P \arrow[r, "\mu"] \arrow[d, dash] & P' \arrow[d, dash, dashed] \\ Q \arrow[r, dashed, "\mu"] & Q' \end{tikzcd}$$

Bisimulations in the pi calculus

The main differences is the way they handle substitutions

$$
\begin{tikzcd}
P \arrow[r, "a(x)"] \arrow[d, dash] & P' \arrow[d, dash, dashed] \\
Q \arrow[r, dashed, "a(x)"] & Q'
\end{tikzcd}
$$

Ground

Bisimulations in the pi calculus

The main differences is the way they handle substitutions

$$
\begin{tikzcd}
P \arrow[r, "a(x)"] \arrow[d, no head] & P' & P'\{^u/_x\} \arrow[d, dash, dashed] \\
Q \arrow[r, dashed, "a(x)"] & Q' & Q'\{^u/_x\}
\end{tikzcd}
$$

for any name u

Early/Late

Bisimulations in the pi calculus

The main differences is the way they handle substitutions

$$\begin{tikzcd} P \arrow[d, no head] & P\sigma \arrow[r, "\mu"] & P' \arrow[d, dashed, no head] \\ Q & Q\sigma \arrow[r, dashed, "\mu"] & Q' \end{tikzcd}$$

for any σ

Open

Bisimulations in the pi calculus

The main differences is the way they handle substitutions

$$
\begin{tikzcd}
P \arrow[d, dash, "D"] & P\sigma \arrow[r, "\mu"] & P' \arrow[d, dash, dashed, "D'"] \\
Q & Q\sigma \arrow[r, dashed, "\mu"] & Q'
\end{tikzcd}
$$

for any σ that respects D

Open

Distinctions D to prevent from fusing previously extruded names with free names.

Bisimulations in the pi calculus

The main differences is the way they handle substitutions

$$\begin{tikzcd} P \arrow[d, dash, "D"] & P\sigma \arrow[r, "\mu"] & P' \arrow[d, dashed, dash, "D'"] \\ Q & Q\sigma \arrow[r, dashed, "\mu"] & Q' \end{tikzcd}$$

for any σ that respects D

Open

The quantification over all substitutions gives a *call-by-need* flavor to the bisimulation. This idea is exploited by the tools which needs to inspect only *most general unifiers*.

$$\text{O-COMM-L}\ \frac{P \xrightarrow[M]{a(x)} P' \qquad Q \xrightarrow[N]{\overline{b}\,u} Q'}{P \mid Q \xrightarrow[MN[a=b]]{\tau} P'\{^{u}/_{x}\} \mid Q'}$$

Bisimulations in spi calculus

- Consider $P(M) := (\nu k)\, \overline{c}\langle \mathrm{Enc}^{\mathrm{s}}_{k} M \rangle . \mathbf{0}$.
  We want $P(M) \approx P(N)$ since k is private and never revealed.

Bisimulations in spi calculus

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  We want $P(M) \approx P(N)$ since k is private and never revealed.

$$\begin{array}{ccc} P(M) & \text{———} & P(N) \\ \Big\downarrow{\scriptstyle \overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_{k} M} & & \\ \mathbf{0} & & \mathbf{0} \end{array}$$

Bisimulations in spi calculus

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$$
\begin{tikzcd}
P(M) \arrow[r, dash] \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_k M"'] & P(N) \arrow[d, dashed, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_k N"] \\
\mathbf{0} & \mathbf{0}
\end{tikzcd}
$$

Bisimulations in spi calculus

- Consider $P(M) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{\mathrm{s}}_{k} M\rangle.\,\mathbf{0}$.
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\mathbf{0} & \mathbf{0}
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- Bisimulations of the pi calculus are too fine-grained.

Bisimulations in spi calculus

- Consider $P(M) := (\nu k)\,\overline{c}\langle \mathsf{Enc}^{\mathrm{s}}_{k} M\rangle.\,\mathbf{0}$.
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P(M) \arrow[r, dash] \arrow[d, "\overline{c}\,(\nu k)\mathsf{Enc}^{\mathrm{s}}_{k} M"'] & P(N) \arrow[d, dashed, "\overline{c}\,(\nu k)\mathsf{Enc}^{\mathrm{s}}_{k} N"] \\
\mathbf{0} & \mathbf{0}
\end{tikzcd}$$

- Bisimulations of the pi calculus are too fine-grained.
- Some pair of messages should be indistinguishable.
- Bisimulations are extended with a data structure that represents the observer knowledge. This has led to various notions of *environment-sensitive* bisimulations (framed, alley, hedged, ...)

Hedged bisimulation def.

Borgström and Nestmann.

Hedge

A hedge $h \in \boldsymbol{H}$ is a finite set of pairs of messages.
Intuitively $(M, N) \in h$ means that M and N are indistinguishable.

A hedged bisimulation relates triples (h, P, Q).

Hedged bisimulation def.

Borgström and Nestmann.

$$P(M) := (\nu k)\, \overline{c}\langle \mathrm{Enc}^{\mathrm{s}}_{k} M \rangle . \mathbf{0}$$

$$P(M) \qquad (c, c) \qquad P(N)$$

Hedged bisimulation def.

Borgström and Nestmann.

$$P(M) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{s}_{k} M\rangle.\,\mathbf{0}$$

$$\begin{array}{ccc} P(M) & (c, c) & P(N) \\ \big\downarrow{\scriptstyle \overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} M} & & \\ \mathbf{0} & & \end{array}$$

Hedged bisimulation def.

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$$P(M) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{s}_{k} M\rangle.\,\mathbf{0}$$

$$\begin{tikzcd}
P(M) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} M"'] & (c, c) & P(N) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} N"] \\
\mathbf{0} & & \mathbf{0}
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$$P(M) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{\mathrm{s}}_{k} M\rangle.\,\mathbf{0}$$

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P(M) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_{k} M"'] & (c, c) & P(N) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_{k} N"] \\
\mathbf{0} & (\mathrm{Enc}^{\mathrm{s}}_{k} M, \mathrm{Enc}^{\mathrm{s}}_{k} N) & \mathbf{0}
\end{tikzcd}$$

Hedged bisimulation (def.)

Borgström and Nestmann.

$$Q(M,N) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{s}_{k} M\rangle.\overline{c}\langle \mathrm{Enc}^{s}_{k} N\rangle.\,\mathbf{0}$$

$$\begin{array}{ccc}
Q(M,M) & (c,c) & Q(M,N) \\
\Big\downarrow{\scriptstyle \overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} M} & & \Big\downarrow{\scriptstyle \overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} M} \\
 & (\mathrm{Enc}^{s}_{k} M, \mathrm{Enc}^{s}_{k} M) & \\
\Big\downarrow{\scriptstyle \overline{c}\,\mathrm{Enc}^{s}_{k} M} & & \Big\downarrow{\scriptstyle \overline{c}\,\mathrm{Enc}^{s}_{k} N} \\
\mathbf{0} & (\mathrm{Enc}^{s}_{k} M, \mathrm{Enc}^{s}_{k} N) & \mathbf{0}
\end{array}$$

Hedged bisimulation def.

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$$\begin{tikzcd}
Q(M,M) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_k M"'] & (c,c) & Q(M,N) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_k M"] \\
{} \arrow[d, "\overline{c}\,\mathrm{Enc}^{\mathrm{s}}_k M"'] & ({\color{blue}\mathrm{Enc}^{\mathrm{s}}_k M}, {\color{red}\mathrm{Enc}^{\mathrm{s}}_k M}) & {} \arrow[d, "\overline{c}\,\mathrm{Enc}^{\mathrm{s}}_k N"] \\
\mathbf{0} & ({\color{blue}\mathrm{Enc}^{\mathrm{s}}_k M}, {\color{green}\mathrm{Enc}^{\mathrm{s}}_k N}) & \mathbf{0}
\end{tikzcd}$$

Hedged bisimulation def.

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$$Q(M,N) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{\mathrm{s}}_k M\rangle.\overline{c}\langle \mathrm{Enc}^{\mathrm{s}}_k N\rangle.\,\mathbf{0}$$

$$\begin{array}{ccc}
Q(M,M) & (c,c) & Q(M,N) \\
\downarrow{\scriptstyle \overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_k M} & & \downarrow{\scriptstyle \overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_k M} \\
 & ({\color{blue}\mathrm{Enc}^{\mathrm{s}}_k M}, {\color{red}\mathrm{Enc}^{\mathrm{s}}_k M}) & \\
\downarrow{\scriptstyle \overline{c}\,\mathrm{Enc}^{\mathrm{s}}_k M} & & \downarrow{\scriptstyle \overline{c}\,\mathrm{Enc}^{\mathrm{s}}_k N} \\
\mathbf{0} & ({\color{blue}\mathrm{Enc}^{\mathrm{s}}_k M}, {\color{green}\mathrm{Enc}^{\mathrm{s}}_k N}) & \mathbf{0}
\end{array}$$

The hedge must be consistent def..

$$O := c(x).c(y).[x = y]\overline{c}\langle \mathrm{fail}\rangle.\,\mathbf{0}$$

Hedged bisimulation def.

Borgström and Nestmann.

$$R(M) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{s}_{k} M\rangle.\overline{c}\langle k\rangle.\,\mathbf{0}$$

$$\begin{array}{ccc}
R(M) & (c, c) & R(N) \\
\big\downarrow \overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} M & & \big\downarrow \overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} N \\
 & (\mathrm{Enc}^{s}_{k} M, \mathrm{Enc}^{s}_{k} N) & \\
\big\downarrow \overline{c}\,k & & \big\downarrow \overline{c}\,k \\
\mathbf{0} & (k, k) & \mathbf{0}
\end{array}$$

Hedged bisimulation def.

Borgström and Nestmann.

$$R(M) := (\nu k)\, \overline{c}\langle \mathrm{Enc}^{\mathrm{s}}_{k} M\rangle . \overline{c}\langle k\rangle .\, \mathbf{0}$$

$$\begin{tikzcd}
R(M) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_{k} M"'] & (c, c) & R(N) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{\mathrm{s}}_{k} N"] \\
{} \arrow[d, "\overline{c}\, k"'] & (\mathrm{Enc}^{\mathrm{s}}_{k} M, \mathrm{Enc}^{\mathrm{s}}_{k} N) & {} \arrow[d, "\overline{c}\, k"] \\
\mathbf{0} & \begin{array}{c}(k, k)\\ (M, N)\end{array} & \mathbf{0}
\end{tikzcd}$$

The hedge is analysed after outputs def..

Hedged bisimulation def.

Borgström and Nestmann.

$$S_1(M) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{s}_{k} M\rangle.c(x).[x=k]\overline{c}\langle k\rangle.\mathbf{0}$$
$$S_2(M) := (\nu k)\,\overline{c}\langle \mathrm{Enc}^{s}_{k} M\rangle.c(x).\mathbf{0}$$

$$\begin{tikzcd}
S_1(M) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} M"'] & (c,c) & S_2(M) \arrow[d, "\overline{c}\,(\nu k)\mathrm{Enc}^{s}_{k} M"] \\
{} \arrow[d, "c(x)"'] & (\mathrm{Enc}^{s}_{k} M, \mathrm{Enc}^{s}_{k} M) & {} \arrow[d, "c(x)"] \\
{[x=k]\mathbf{0}} & & \mathbf{0}
\end{tikzcd}$$

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\cdot \arrow[d, "c(x)"'] & (\mathrm{Enc}^{s}_{k} M, \mathrm{Enc}^{s}_{k} M) & \cdot \arrow[d, "c(x)"] \\
{[x=k]\,\mathbf{0}} & & \mathbf{0}
\end{tikzcd}$$

The possible pairs of input messages are constructed using the current knowledge and possibly some *fresh names* def.

Open hedged bisimulation def.

Delaying instantiation of input variables

- Which names are subjects to substitutions?
  - Input variables.
- What are the possible objects of substitutions?
  - Messages constructed using the knowledge available at the moment of the input and possibly some fresh names.
- A variable dynamically typed as a name is not replaced by a compound message LTS.

Open hedged bisimulation def.

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Hence the form of S-environments $se = (h, v, \prec, (\gamma_l, \gamma_r))$.

Open hedged bisimulation def.

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Hence the form of S-environments $se = (h, v, \prec, (\gamma_l, \gamma_r))$.

consistency of S-environments

A S-environment is consistent if for any instantiation of input variables, the resulting hedge is consistent.

Symbolic characterisation

- Relies on the definition of a *symbolic LTS* def.
- The idea is to record —without checking— the conditions needed to enable a transition.

$$P \overset{\mu}{\underset{\Phi}{\longmapsto}} P'$$

- The symbolic LTS helps to characterise precisely the set of substitutions σ such that $P\sigma \xrightarrow{\mu} P'$.
- Given a symbolic transition $P \overset{\mu}{\underset{\Phi}{\longmapsto}} P'$, there is a finite complete set of solutions of Φ.

Outline

1. From protocol narrations to spi calculus
2. An open variant of bisimulation for the spi calculus
3. A formalization in Coq

Representation of binders

de Bruijn indices

Representation of $a(x).[\mathrm{Dec}^{s}_{k}\, x : \boldsymbol{M}](\nu l)\, \overline{b}\langle l\rangle.\,\mathbf{0}$

$$z\ y\ x\ \cdots\ l\ k\ j\ \cdots\ c\ b\ a$$

$$0\ \lambda.\ [\mathrm{Dec}^{s}_{11}\ 0 : \boldsymbol{M}]\ \nu\ \overline{3}\langle 0\rangle.\mathbf{0}$$

Representation of binders

de Bruijn indices

Several operations have to be defined to handle de Bruijn indices. more

Example: $\mathrm{lift}_d(k, t)$ makes room for k new binders in t

$$z\ y\ x\ \cdots\ c\ b\ a$$

$$\lambda.\overline{1}\langle 0\rangle.\mathbf{0}$$

$$z\ y\ x\ \cdots\ c\ b\ a$$

$$0\ \lambda.\overline{0}\langle 24\rangle.\mathbf{0}$$

$$z\ y\ x\ \cdots\ c\ b\ a$$

$$\lambda.(\overline{1}\langle 0\rangle.\mathbf{0} \mid 1\ \lambda.\overline{0}\langle 25\rangle.\mathbf{0})$$

Representation of binders

de Bruijn indices

In practise:

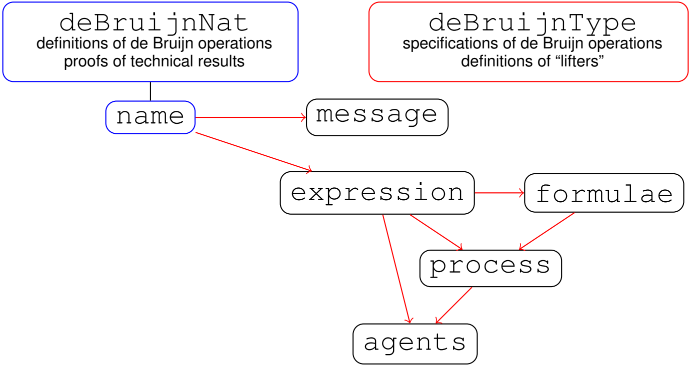
- 5 operations on indices, 6 types (names, messages, ...)
- about 60 useful facts relate these operations
- not scalable and tedious to define and prove several times the same operations/facts

Representation of binders

de Bruijn indices

Instead

1. define on names
2. lift to other types



Abstracting the labelled transition system

- There are several LTS to define.
- Some properties are shared
(e.g. structural congruence preserves the transitions)
- These LTS all follow the same pattern.
- Instead of defining each LTS separately, we make a functor and thus defer the definition of the semantics to the definitions of the semantics of actions.

Abstracting the labelled transition system

We rely on a set of actions $\mathcal{A}$ and several functions to manipulate them:

- mkSil : $\mathcal{A}$ (silent)
- mkInp : $\boldsymbol{E} \rightarrow \mathcal{A} \cup \{\bot\}$ (input)
- mkOutp : $\boldsymbol{E} \times \boldsymbol{E} \rightarrow (\mathcal{A} \times \boldsymbol{E}) \cup \{\bot\}$ (output)
- mkRes : $\mathcal{A} \rightarrow \mathcal{A} \cup \{\bot\}$ (restriction)
- mkIf : $\boldsymbol{F} \times \mathcal{A} \rightarrow \mathcal{A} \cup \{\bot\}$ (guard)
- mkInt : $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \cup \{\bot\}$ (interact)

Abstracting the labelled transition system

We then define a parametrised LTS.

$$\textsc{Input}\ \frac{\mathsf{mkInp}(E) = \alpha \in \mathcal{A}}{E\lambda.P \xrightarrow{\alpha} \lambda.P} \qquad \textsc{Output}\ \frac{\mathsf{mkOutp}(E, F) = (\alpha, M) \in \mathcal{A} \times \boldsymbol{E}}{\overline{E}\langle F\rangle.P \xrightarrow{\alpha} \langle M\rangle P}$$

$$\textsc{Close-L}\ \frac{P \xrightarrow{\alpha} F \qquad Q \xrightarrow{\beta} C \qquad \mathsf{mkInt}(\alpha, \beta) = \gamma \in \mathcal{A}}{P \mid Q \xrightarrow{\gamma} F \bullet C}$$

Overview of the formalization

- Monadic pi calculus
- Pi LTS
- Spi calculus
- Hedges and their properties
- Spi LTS: standard, with type constraints, symbolic and their properties
- Crash test: result about structural congruence
- Late hedged bisimulation, correctness of up-to techniques
- Small examples of bisimulations

Conclusion

- A formal semantics for protocol narrations.
  - A rigorous translation into spi calculus.
- An open style definition of bisimulation for the spi calculus.
  - It is a sound proof technique.
  - It is an extension of open bisimulation of the pi calculus.
  - Its projection down to the pi calculus has enabled us to better understand the original notion of open bisimulation.
  - A symbolic characterisation as a promising first step towards mechanisation.
- A formalization in a proof assistant.
  - Very useful while elaborating the theory.
  - Already a framework to reason formally about cryptographic protocols in Coq.

Future work

- Study furthermore open hedged bisimilarity.
  - Congruence properties.
  - Mechanisation.
- Complete the formalization in Coq.
  - Realise the dream of having a correct-by-construction equivalence checker for the spi calculus.
  - Define smart tactics for reasoning directly in Coq (e.g. interface with the tool that handles the decidable fragment)

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  - Define smart tactics for reasoning directly in Coq (e.g. interface with the tool that handles the decidable fragment)
- Demos?

The end.

The spi calculus back

Syntax

- Countably infinite set of *names*.
  Communication channels, nonces, atomic data, ...
- Messages

$$M, N \quad ::= \quad x \mid (M\,.\,N) \mid \mathrm{Enc}^{\mathrm{s}}_{N} M$$

- Expressions

$$\begin{array}{rcl} E, F & ::= & x \mid (E\,.\,F) \mid \mathrm{Enc}^{\mathrm{s}}_{F} E \\ & \mid & \pi_1\,(E) \mid \pi_2\,(E) \mid \mathrm{Dec}^{\mathrm{s}}_{F} E \end{array}$$

- Guards

$$\phi \quad ::= \quad [\,E = F\,] \mid [\,E : \mathcal{N}\,]$$

Syntax back

continued

- Processes

$$\begin{array}{rcl} P, Q & ::= & \mathbf{0} \mid E(x).P \mid \overline{E}\langle F\rangle.P \\ & \mid & \phi P \mid (\nu x)\, P \\ & \mid & P \,|\, Q \mid P + Q \mid\; !\, P \end{array}$$

- Agents

$$\begin{array}{rcl} A & ::= & P \\ & \mid & (x)P \\ & \mid & (\nu \tilde{z})\,\langle M\rangle P \quad \text{where } \{\tilde{z}\} \subseteq \mathrm{n}(M) \end{array}$$

Labelled transitions system back

Late semantics

$$\textsc{Input}\ \frac{\mathbf{e}_{\mathrm{c}}(E) = a \in \mathcal{N}}{E(x).P \xrightarrow{a} (x)P} \qquad \textsc{Output}\ \frac{\mathbf{e}_{\mathrm{c}}(E) = a \in \mathcal{N} \qquad \mathbf{e}_{\mathrm{c}}(F) = M \in \boldsymbol{M}}{\overline{E}\langle F\rangle.P \xrightarrow{\overline{a}} \langle M\rangle P}$$

$$\textsc{Close-L}\ \frac{P \xrightarrow{a} F \qquad Q \xrightarrow{\overline{a}} C}{P \mid Q \xrightarrow{\tau} F \bullet C} \qquad \textsc{IfThen}\ \frac{P \xrightarrow{\mu} P'}{\phi P \xrightarrow{\mu} P'}\ \mathbf{e}(\phi) = \mathbf{true}$$

$$\textsc{Res}\ \frac{P \xrightarrow{\mu} A}{(\nu z)\, P \xrightarrow{\mu} (\boldsymbol{\nu} z)\, A}\ z \notin \mathrm{n}(\mu) \qquad \textsc{Par-L}\ \frac{P \xrightarrow{\mu} A}{P \mid Q \xrightarrow{\mu} A \mid Q}$$

+ SUM, REP- et ALPHA.

Evaluation of expressions and guards back

- Expressions:

$$
\begin{array}{rcll}
\mathbf{e}_{\mathrm{c}}(a) & := & a & \\
\mathbf{e}_{\mathrm{c}}(\mathrm{Enc}^{\mathrm{s}}_{F}E) & := & \mathrm{Enc}^{\mathrm{s}}_{N}M & \text{if } \mathbf{e}_{\mathrm{c}}(E) = M \in \boldsymbol{M} \\
 & & & \text{and } \mathbf{e}_{\mathrm{c}}(F) = N \in \boldsymbol{M} \\
\mathbf{e}_{\mathrm{c}}((E_1 \,.\, E_2)) & := & (M_1 \,.\, M_2) & \text{if } \mathbf{e}_{\mathrm{c}}(E_1) = M_1 \in \boldsymbol{M} \\
 & & & \text{and } \mathbf{e}_{\mathrm{c}}(E_2) = M_2 \in \boldsymbol{M} \\
\mathbf{e}_{\mathrm{c}}(\mathrm{Dec}^{\mathrm{s}}_{F}E) & := & M & \text{if } \mathbf{e}_{\mathrm{c}}(E) = \mathrm{Enc}^{\mathrm{s}}_{N}M \in \boldsymbol{M} \\
 & & & \text{and } \mathbf{e}_{\mathrm{c}}(F) = N \in \boldsymbol{M} \\
\mathbf{e}_{\mathrm{c}}(\pi_1(E)) & := & M_1 & \text{if } \mathbf{e}_{\mathrm{c}}(E) = (M_1 \,.\, M_2) \in \boldsymbol{M} \\
\mathbf{e}_{\mathrm{c}}(\pi_2(E)) & := & M_2 & \text{if } \mathbf{e}_{\mathrm{c}}(E) = (M_1 \,.\, M_2) \in \boldsymbol{M} \\
\mathbf{e}_{\mathrm{c}}(E) & := & \perp & \text{otherwise}
\end{array}
$$

- Guards:

$$
\begin{array}{rcll}
\mathbf{e}([E = F]) & := & \mathbf{true} & \text{si } \mathbf{e}_{\mathrm{c}}(E) = \mathbf{e}_{\mathrm{c}}(F) = M \in \boldsymbol{M} \\
\mathbf{e}([E : \mathcal{N}]) & := & \mathbf{true} & \text{si } \mathbf{e}_{\mathrm{c}}(E) = a \in \mathcal{N} \\
\mathbf{e}(\phi) & := & \mathbf{false} & \text{otherwise}
\end{array}
$$

Late hedged bisimulation back

A symmetric consistent hedged relation $\mathcal{R}$ is a *(strong) late hedged bisimulation* if whenever $(h, P, Q) \in \mathcal{R}$, we have that

1. if $P \xrightarrow{\tau} P'$ then
   there exists Q' such that $Q \xrightarrow{\tau} Q'$ and $(h, P', Q') \in \mathcal{R}$
2. if $P \xrightarrow{a} (x)P'$ (with $x \notin \mathrm{n}(\pi_1(h))$)
   and $(a, b) \in h$ then
   there exist y and Q' such that $Q \xrightarrow{b} (y)Q'$ (with $y \notin \mathrm{n}(\pi_2(h))$)
   and for all B and (M, N) such that $h \vdash_B (M, N)$
   we have $(h \cup B, P'\{^M/_x\}, Q'\{^N/_y\}) \in \mathcal{R}$.
3. if $P \xrightarrow{\overline{a}} (\nu\tilde{c})\,\langle M\rangle P'$ (with $\{\tilde{c}\} \cap \mathrm{n}(\pi_1(h)) = \emptyset$)
   and $(a, b) \in h$ then
   there exist $\tilde{d}$, Q' and N such that $Q \xrightarrow{\overline{b}} (\nu\tilde{d})\,\langle N\rangle Q'$
   (with $\left\{\tilde{d}\right\} \cap \mathrm{n}(\pi_2(h)) = \emptyset$)
   and $(\mathcal{I}(h \cup \{(M, N)\}), P', Q') \in \mathcal{R}$.

Synthesis of a hedge and possible inputs back

Synthesis of a hedge

The synthesis $\mathcal{S}(h)$ is the smallest set that satisfies

$$\textsc{Syn-Inc}\ \frac{(M, N) \in h}{(M, N) \in \mathcal{S}(h)}$$

$$\textsc{Syn-Enc-s}\ \frac{(M_1, N_1) \in \mathcal{S}(h) \qquad (M_2, N_2) \in \mathcal{S}(h)}{(\mathrm{Enc}^{\mathrm{s}}_{M_2} M_1, \mathrm{Enc}^{\mathrm{s}}_{N_2} N_1) \in \mathcal{S}(h)}$$

$$\textsc{Syn-Pair}\ \frac{(M_1, N_1) \in \mathcal{S}(h) \qquad (M_2, N_2) \in \mathcal{S}(h)}{((M_1 \,.\, M_2), (N_1 \,.\, N_2)) \in \mathcal{S}(h)}$$

Synthesis of a hedge and possible inputs back

Possible inputs

Let $h \in \boldsymbol{H}$, $(M, N) \in \boldsymbol{M} \times \boldsymbol{M}$
Let $B \subseteq \mathcal{N} \times \mathcal{N}$ a consistent hedge such that

- $\pi_1(B) \cap \mathrm{n}(\pi_1(h)) = \emptyset$
- $\pi_2(B) \cap \mathrm{n}(\pi_2(h)) = \emptyset$

i.e. the names of B are fresh component-wise w.r.t. those of h.
We write $h \vdash_B (M, N)$ if

- $\forall (b_1, b_2) \in B : b_1 \in \mathrm{n}(M) \vee b_2 \in \mathrm{n}(N)$
- $(M, N) \in \mathcal{S}(h \cup B)$

Analysis of a hedge and irreducibles back

Analysis

The analysis $\mathcal{A}(h)$ is the smallest hedge that is closed by $\mathrm{analz}(\cdot)$.

$$\textsc{Ana-Inc}\ \frac{(M, N) \in h}{(M, N) \in \mathrm{analz}(h)}$$

$$\textsc{Ana-Dec-s}\ \frac{(\mathsf{Enc}^{\mathrm{s}}_{M_2} M_1, \mathsf{Enc}^{\mathrm{s}}_{N_2} N_1) \in \mathrm{analz}(h) \qquad (M_2, N_2) \in \mathcal{S}(h)}{(M_1, N_1) \in \mathrm{analz}(h)}$$

$$\textsc{Ana-Fst}\ \frac{((M_1 \,.\, M_2), (N_1 \,.\, N_2)) \in \mathrm{analz}(h)}{(M_1, N_1) \in \mathrm{analz}(h)}$$

$$\textsc{Ana-Snd}\ \frac{((M_1 \,.\, M_2), (N_1 \,.\, N_2)) \in \mathrm{analz}(h)}{(M_2, N_2) \in \mathrm{analz}(h)}$$

Analysis of a hedge and irreducibles back

Irreducibles

$\mathcal{I}(h)$ is the smallest hedge such that $\mathcal{S}(\mathcal{I}(h)) = \mathcal{S}(\mathcal{A}(h))$.

Definition

A hedge h is irreducible iff $\mathcal{I}(h) = h$.

Consistency of a hedge back

Consistency

A hedge h is consistent iff:
Whenever $(M, N) \in h$

- $M \in \mathcal{N} \iff N \in \mathcal{N}$
- whenever $(M', N') \in h$: $M = M' \iff N = N'$
- $M \neq (M_1 \,.\, M_2)$ and $N \neq (N_1 \,.\, N_2)$
- if $M = \mathrm{Enc}^{\mathrm{s}}_{M_2} M_1$ then $(M_2, N_2) \notin \mathcal{S}(h)$
- if $N = \mathrm{Enc}^{\mathrm{s}}_{N_2} N_1$ then $(M_2, N_2) \notin \mathcal{S}(h)$

Lemma

A consistent hedge is irreducible.

S-environments back

Definition (S-environment)

A S-environment is a quadruple $se = (h, v, \prec, (\gamma_l, \gamma_r))$ where $h \in \boldsymbol{H}$, $v \subseteq \mathcal{N} \times \mathcal{N}$ is a consistent hedge, $\prec \subseteq h \times v$, $\gamma_l \subseteq \pi_1(v)$ and $\gamma_r \subseteq \pi_2(v)$.

Hedge available

The *hedge available* to $(x, y) \in v$ according to $\prec$ is defined by $se|_{(x,y)} := \{(M, N) \in h \mid (M, N) \prec (x, y)\}$.

Concrete hedge

The *concrete hedge* of *se* is $\mathfrak{H}(se) := h \cup v$.

Respectful substitutions back

Definition (Respectful substitutions)

Let (σ, ρ) be a pair of substitutions, $B \subseteq \mathcal{N} \times \mathcal{N}$ a consistent hedge and $se = (h, v, \prec, (\gamma_l, \gamma_r))$ a S-environment. We say that (σ, ρ) *respects se with B* — written $(\sigma, \rho) \rhd_B se$ — if

1. $\mathrm{supp}(\sigma) \subseteq \pi_1(v)$
2. $\mathrm{supp}(\rho) \subseteq \pi_2(v)$
3. $\forall (b_1, b_2) \in B : b_1 \in \mathrm{n}(\sigma(\pi_1(v))) \vee b_2 \in \mathrm{n}(\rho(\pi_2(v)))$
4. $\pi_1(B) \cap (\mathrm{n}(\pi_1(h)) \setminus \pi_1(v)) = \emptyset$
5. $\pi_2(B) \cap (\mathrm{n}(\pi_2(h)) \setminus \pi_2(v)) = \emptyset$
6. $\forall (x, y) \in v : (x\sigma, y\rho) \in \mathcal{S}(\mathcal{I}(se|_{(x,y)}(\sigma, \rho) \cup B))$
7. $\forall x \in \gamma_l : x\sigma \in \mathcal{N}$
8. $\forall y \in \gamma_r : y\rho \in \mathcal{N}$

Open hedged bisimulation back

A symmetric consistent open hedged relation $\mathcal{R}$ is an *open hedged bisimulation* if for all $(se, P, Q) \in \mathcal{R}$, for all σ, ρ and B such that $(\sigma, \rho) \triangleright_B se$,

internal communications

if $P\sigma \xhookrightarrow[S_1]{\tau} P'$ then

there exist Q' and S_2 such that $Q\rho \xhookrightarrow[S_2]{\tau} Q'$

and $(se_B^{(\sigma,\rho)} +_c (S_1, S_2), P', Q') \in \mathcal{R}$

Open hedged bisimulation back

A symmetric consistent open hedged relation $\mathcal{R}$ is an *open hedged bisimulation* if for all $(se, P, Q) \in \mathcal{R}$, for all σ, ρ and B such that $(\sigma, \rho) \triangleright_B se$,

inputs

if $P\sigma \xhookrightarrow[S_1]{a} (x)P'$ (with $x \notin \mathrm{n}(\pi_1(\mathfrak{H}(se_B^{(\sigma,\rho)}))))$
and $(a, b) \in \mathcal{S}(\mathcal{I}(\mathfrak{H}(se_B^{(\sigma,\rho)})))$ then
there exist y, Q' and S_2 such that $Q\rho \xhookrightarrow[S_2]{b} (y)Q'$ (with $y \notin \mathrm{n}(\pi_2(\mathfrak{H}(se_B^{(\sigma,\rho)}))))$
and $(se_B^{(\sigma,\rho)} +_{\mathrm{i}} (x, y) +_{\mathrm{c}} (S_1, S_2), P', Q') \in \mathcal{R}$

Open hedged bisimulation back

A symmetric consistent open hedged relation $\mathcal{R}$ is an *open hedged bisimulation* if for all $(se, P, Q) \in \mathcal{R}$, for all σ, ρ and B such that $(\sigma, \rho) \triangleright_B se$,

outputs

if $P\sigma \xrightarrow[S_1]{\overline{a}} (\nu\tilde{c})\,\langle M\rangle P'$ (with $\{\tilde{c}\} \cap \mathrm{n}(\pi_1(\mathfrak{H}(se_B^{(\sigma,\rho)}))) = \emptyset$)
and $(a, b) \in \mathcal{S}(\mathcal{I}(\mathfrak{H}(se_B^{(\sigma,\rho)})))$ then
there exist $\tilde{d}$, N, Q' and S_2 such that $Q\rho \xrightarrow[S_2]{\overline{b}} (\nu\tilde{d})\,\langle N\rangle Q'$
(with $\left\{\tilde{d}\right\} \cap \mathrm{n}(\pi_2(\mathfrak{H}(se_B^{(\sigma,\rho)}))) = \emptyset$)
and $(se_B^{(\sigma,\rho)} +_{\mathrm{o}} (M, N) +_{\mathrm{c}} (S_1, S_2), P', Q') \in \mathcal{R}$

A LTS that collects type constraints back

$$\text{NC-SILENT}\ \frac{}{\tau.P \xhookrightarrow[\emptyset]{\tau} P}$$

$$\text{NC-INPUT}\ \frac{\mathbf{e}_c(E) = a \in \mathcal{N}}{E(x).P \xhookrightarrow[\{a\}]{a} (x)P}$$

$$\text{NC-OUTPUT}\ \frac{\mathbf{e}_c(E) = a \in \mathcal{N} \qquad \mathbf{e}_c(F) = M \in \boldsymbol{M}}{\overline{E}\langle F\rangle.P \xhookrightarrow[\{a\}]{\overline{a}} \langle M\rangle P}$$

$$\text{NC-IFTHEN}\ \frac{P \xhookrightarrow[S]{\mu} A}{\phi P \xhookrightarrow[S \cup \mathbf{nc}(\phi)]{\mu} A}\ \mathbf{e}(\phi) = \mathbf{true}$$

where $\mathbf{nc}([E:\mathcal{N}]) := \{\mathbf{e}_c(E)\}$ and $\mathbf{nc}([E=F]) := \emptyset$.

Properties back

Theorem

The two semantics are equivalent:

1. *If* $P \xrightarrow{\mu} A$ *there exists* $S \subseteq \mathcal{N}$ *such that* $P \underset{S}{\overset{\mu}{\hookrightarrow}} A$.
2. *If* $P \underset{S}{\overset{\mu}{\hookrightarrow}} A$ *then* $P \xrightarrow{\mu} A$.

Lemma

If $P \underset{S}{\overset{\mu}{\hookrightarrow}} A$ *and* $\sigma : \mathcal{N} \to \mathbf{M}$ *is a substitution such that* $S\sigma \subseteq \mathcal{N}$ *then* $P\sigma \underset{S\sigma}{\overset{\mu\sigma}{\hookrightarrow}} A\sigma$.

A symbolic LTS back

$$\text{S-GUARD}\ \frac{P \overset{\mu}{\underset{c}{\longmapsto}} A}{\phi P \overset{\mu}{\underset{c\&\{\phi\}}{\longmapsto}} A} \qquad \text{S-INPUT}\ \frac{}{E(x).P \overset{\mathbf{e}_{\mathrm{a}}(E)}{\underset{\{[E:\mathcal{N}]\}}{\longmapsto}} (x)P}$$

$$\text{S-OUTPUT}\ \frac{}{\overline{E}\langle F\rangle.P \overset{\overline{\mathbf{e}_{\mathrm{a}}(E)}}{\underset{\{[E:\mathcal{N}],[F:\boldsymbol{M}]\}}{\longmapsto}} \langle \mathbf{e}_{\mathrm{a}}(F)\rangle P}$$

$$\text{S-CLOSE-L}\ \frac{P \overset{E}{\underset{c_1}{\longmapsto}} F \qquad Q \overset{\overline{E'}}{\underset{c_2}{\longmapsto}} C}{P \mid Q \overset{\tau}{\underset{\{[E=E']\}\&c_1\&c_2}{\longmapsto}} F \bullet C}$$

$$\text{S-RES}\ \frac{P \overset{\mu}{\underset{c}{\longmapsto}} A}{(\nu z)\, P \overset{\mu}{\underset{\nu_+(z,c)}{\longmapsto}} (\boldsymbol{\nu} z)\, A}\ z \notin \mathrm{n}(\mu)$$

Transition constraints back

- A transition constraint has the form $(\nu\tilde{z})\,\Phi$ where Φ is a finite set of guards and $\tilde{z}$ is a finite set of names that occur in Φ, i.e. $\{\tilde{z}\} \subseteq \mathrm{n}(\Phi)$
- Composition of constraints:
  - Conjunction of $c_1 = (\nu\tilde{z}_1)\,\Phi_1$ and $c_2 = (\nu\tilde{z}_2)\,\Phi_2$ where $\{\tilde{z}_1\} \cap \{\tilde{z}_2\} = \emptyset$, $\{\tilde{z}_1\} \cap \mathrm{fn}(c_2) = \{\tilde{z}_2\} \cap \mathrm{fn}(c_1) = \emptyset$

$$c_1 \,\&\, c_2 := (\nu\tilde{z}_1\tilde{z}_2)\,(\Phi_1 \cup \Phi_2)$$

  - Restriction of name x.
    If $c = (\nu\tilde{z})\,\Phi$ and $x \notin \{\tilde{z}\}$:

$$\begin{array}{rcll} \nu_+(x, c) & := & (\nu x\tilde{z})\,\Phi & \text{if } x \in \mathrm{fn}(c) \\ & := & c & \text{otherwise} \end{array}$$

Abstract evaluation back

Abstract evaluation of expressions:

$$
\begin{array}{rcll}
\mathbf{e}_{\mathrm{a}}(a) & := & a & \text{if } a \in \mathcal{N} \\
\mathbf{e}_{\mathrm{a}}(\mathrm{Enc}^{\mathrm{s}}_{F} E) & := & \mathrm{Enc}^{\mathrm{s}}_{\mathbf{e}_{\mathrm{a}}(F)} \mathbf{e}_{\mathrm{a}}(E) & \\
\mathbf{e}_{\mathrm{a}}((E\,.\,F)) & := & (\mathbf{e}_{\mathrm{a}}(E)\,.\,\mathbf{e}_{\mathrm{a}}(F)) & \\
\mathbf{e}_{\mathrm{a}}(\mathrm{Dec}^{\mathrm{s}}_{F} E) & := & E_1 & \text{if } \mathbf{e}_{\mathrm{a}}(E) = \mathrm{Enc}^{\mathrm{s}}_{E_2} E_1 \\
 & & \mathrm{Dec}^{\mathrm{s}}_{\mathbf{e}_{\mathrm{a}}(F)} \mathbf{e}_{\mathrm{a}}(E) & \text{otherwise} \\
\mathbf{e}_{\mathrm{a}}(\pi_1\,(E)) & := & E_1 & \text{if } \mathbf{e}_{\mathrm{a}}(E) = (E_1\,.\,E_2) \\
 & & \pi_1\,(\mathbf{e}_{\mathrm{a}}(E)) & \text{otherwise} \\
\mathbf{e}_{\mathrm{a}}(\pi_2\,(E)) & := & E_2 & \text{if } \mathbf{e}_{\mathrm{a}}(E) = (E_1\,.\,E_2) \\
 & & \pi_2\,(\mathbf{e}_{\mathrm{a}}(E)) & \text{otherwise}
\end{array}
$$

Properties back

Define $>_{\mathrm{o}}$ as being the smallest precongruence on expressions that satisfies:

- $\pi_1\,((E_1\,.\,E_2)) >_{\mathrm{o}} E_1$ if $\mathbf{e}_{\mathrm{c}}(E_2) \neq \bot$
- $\pi_2\,((E_1\,.\,E_2)) >_{\mathrm{o}} E_2$ if $\mathbf{e}_{\mathrm{c}}(E_1) \neq \bot$
- $\mathrm{Dec}^{\mathrm{s}}_{E_2}\mathrm{Enc}^{\mathrm{s}}_{E_2}E_1 >_{\mathrm{o}} E_1$ if $\mathbf{e}_{\mathrm{c}}(E_2) \neq \bot$

Extend this relation to agents in:

- $A >^{=}_{\mathrm{o}} B$ (A, B are concrete agents)
- $A >^{\mathbf{e}}_{\mathrm{o}} B$ (A is symbolic, B is concrete)

(two ways to handle concretions)

Properties back

continued

Theorem

Let $P, Q \in \mathbf{P}$ and assume that $P >_{\mathrm{o}} Q$.

1. *If $P \overset{\mu}{\underset{S}{\hookrightarrow}} A$ then $Q \overset{\mu}{\underset{S}{\hookrightarrow}} B$ and $A >_{\mathrm{o}}^{=} B$*
2. *If $Q \overset{\mu}{\underset{S}{\hookrightarrow}} B$ then $P \overset{\mu}{\underset{S}{\hookrightarrow}} A$ and $A >_{\mathrm{o}}^{=} B$*

Theorem

Let $P, Q \in \mathbf{P}$ and $\sigma : \mathcal{N} \to \mathbf{M}$ a substitution.

1. *If $P \overset{\mu_s}{\underset{c}{\longmapsto}} A$ and $\mathbf{e}(c\sigma) = \mathbf{true}$ then $P\sigma \xhookrightarrow[\mathbf{nc}(c\sigma)]{\mathbf{e}_{\mathrm{c}}(\mu_s\sigma)} B$ with $A\sigma >_{\mathrm{o}}^{\mathbf{e}} B$*
2. *If $P\sigma \overset{\mu}{\underset{S}{\hookrightarrow}} B$ then $P \overset{\mu_s}{\underset{c}{\longmapsto}} A$ with $\mathbf{e}(c\sigma) = \mathbf{true}$, $\mathbf{nc}(c\sigma) = S$, $\mathbf{e}_{\mathrm{c}}(\mu_s\sigma) = \mu$ and $A\sigma >_{\mathrm{o}}^{\mathbf{e}} B$*

Operations on de Bruijn indices back

- Parametrised by the binding depth d
- $\mathrm{mem}_d(i, t)$ returns **true** iff i is free in t
- $\mathrm{lift}_d(k, t)$ makes room for k new binders in t
  Used in parallel composition of an agent and a process:

$$
\begin{array}{rcl}
(\lambda.P) \mid Q & := & \lambda.(P \mid \mathrm{lift}_0(1, Q)) \\
(\nu^k\langle F\rangle P) \mid Q & := & \nu^k\langle F\rangle(P \mid \mathrm{lift}_0(k, Q))
\end{array}
$$

For instance:

$z\; y\; x\; \cdots c\; b\; a$

$\lambda.\overline{1}\langle 0\rangle.\mathbf{0}$

$z\; y\; x\; \cdots c\; b\; a$

$0\; \lambda.\overline{0}\langle 24\rangle.\mathbf{0}$

$z\; y\; x\; \cdots c\; b\; a$

$\lambda.(\overline{1}\langle 0\rangle.\mathbf{0} \mid 1\; \lambda.\overline{0}\langle 25\rangle.\mathbf{0})$

Operations on de Bruijn indices back

continued

- $\mathsf{swap}_d(k, t)$ makes a circular permutation of the k first indices in t
- $\mathsf{low}_d(t)$ removes the first index
- Used in restriction of an agent:

$$\begin{array}{rcll} \boldsymbol{\nu}(\lambda.P) & := & \lambda.\nu\,\mathsf{swap}_0(1, P) & \\ \boldsymbol{\nu}(\nu^k\langle F\rangle P) & := & \nu^{k+1}\langle F\rangle P & \text{if } \mathsf{mem}_k(0, F) = \textbf{true} \\ & := & \nu^k\langle \mathsf{low}_k(F)\rangle\nu\,\mathsf{swap}_0(k, P) & \text{otherwise} \end{array}$$

- $\mathsf{lsubst}_d(k, \overline{E}, t)$ substitutes the $|\overline{E}|$ first indices with the corresponding expression of $\overline{E}$ in t. The k first indices are bound in $\overline{E}$.

$$(\lambda.P) \bullet (\nu^k\langle F\rangle Q) \quad := \quad \nu^k(\mathsf{lsubst}_0(k, F, P) \mid Q)$$